Solving QCD semiclassically w/ 't Hooft flux

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Spoiler

- Confinement state of 4d YM & QCD can be solved SEMICLASSICALLY on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{w/ 't Hooft flux}}$
  (YT, Ünsal 2201. cf. Yamazaki, Yonekura 1705.05852 (on $\mathbb{R}\times T^3$))

- Confinement vacua of 4d YM have $\mathbb{Z}_N$ discrete label $k \sim k+N$
  (Witten, 't Hooft, Cardy, Rabinovici, ... ; Gaiotto, Kapustin, Komargodski, Seiberg)

- η' (pseudo NG boson of $U(1)_A$) has an extended periodicity $\eta' \sim \eta' + 2\pi N$ by "eating" the discrete label k of confinement vacua.
  (Hayashi, YT, 2402)

θ - parameter

Yang-Mills partition function

$$Z_{YM,\theta} = \int \mathcal{D}a \, e^{-\frac{1}{g^2}\int \mathrm{tr}[|f|^2] + i\theta \underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}[f\wedge f]}_{= Q_{top}}}$$

On each gauge config.,

$$Q_{top}[a] \in \mathbb{Z}$$

$\Rightarrow$ θ is a 2π-periodic parameter

$$Z_{YM,\theta+2\pi} = Z_{YM,\theta}.$$

What does this periodicity mean?

Hamiltonian formalism & θ

Let's take a detour & consider the canonical quantizati (w/ $A_0 = 0$)

$$\mathcal{L}_{\text{Mink}} = \frac{1}{2g^2}\left(\dot{\mathbb{A}}^2 - \mathbb{B}^2\right) + \frac{\theta}{(2\pi)^2}\dot{\mathbb{A}}\cdot\mathbb{B}$$

Conjugate momentum of $\mathbb{A}$:

$$\Pi = \frac{\partial \mathcal{L}_{\text{Mink}}}{\partial \dot{\mathbb{A}}} = \frac{1}{g^2}\dot{\mathbb{A}} + \frac{\theta}{(2\pi)^2}\mathbb{B}$$

Hamiltonian :

$$H_\theta = \dot{\mathbb{A}}\cdot\Pi - \mathcal{L}_{\text{Mink}} = \frac{g^2}{2}\left(\Pi - \frac{\theta}{(2\pi)^2}\mathbb{B}\right)^2 + \frac{1}{2g^2}\mathbb{B}^2 .$$

The canonical quantizati is achieved by $\Pi \to \hat{\Pi} = -\frac{1}{i}\frac{\delta}{\delta \mathbb{A}}$.

Hilbert space is given by the gauge-inv. functional of $\mathbb{A}$:

$$\mathcal{H} = \left\{ \Psi[A] \;\middle|\; \Psi[A^g] = \Psi[A] \text{ for any "small" \& "large" gauge trans. } g \right\}$$

(Inv. under small gauge trans. is the mandatory requirement of the Gauss law.
Large gauge inv. is somewhat a "choice".)

Quantum YM theory is defined by the pair

$$\mathcal{YM}_\theta = (\mathcal{H}, H_\theta) \qquad \left(H_\theta = \frac{g^2}{2}\left(\Pi - \frac{\theta}{2\pi}B\right)^2 + \frac{1}{2g^2}B^2 \right)$$

In this description, $H_{\theta+2\pi} \neq H_\theta$ so $\mathcal{YM}_{\theta+2\pi} \neq \mathcal{YM}_\theta$.

What is the meaning of $\theta \sim \theta + 2\pi$ then?

$$\Leftarrow \quad \mathcal{YM}_{\theta+2\pi} \overset{\text{unitary}}{\sim} \mathcal{YM}_\theta .$$

~~Proof~~

Define the unitary operator

$$U[A] = e^{i\frac{1}{4\pi}\int_{\text{space}} \mathrm{tr}\left[AdA + \frac{2}{3}A^3\right]}$$

Chern-Simons form of the spatial gauge field

(level quantization is requires to have $U[A^g] = U[A]$ so that $U : \mathcal{H} \to \mathcal{H}$.)

then

$$U^{-1}\Pi U = \Pi + \frac{1}{2\pi}B$$

$$\Rightarrow U^{-1}H_{\theta+2\pi}U = H_\theta$$

(BTW, this is true for YM + matters either w/ adjoint and/or fund rep., so far)

Let's assume $\mathcal{H}_\theta$ has the unique ground state Ψ^{GS}_θ.

Since

$$H_{\theta+2\pi} = U H_\theta U^{-1},$$

we find that

$$\Psi^{GS}_{\theta+2\pi}[A] = e^{i\,CS[A]}\,\Psi^{GS}_\theta[A].$$

It's a nontrivial question if $\underbrace{\Psi^{GS}_{\theta+2\pi} \simeq \Psi^{GS}_\theta}_{\text{suggested by DIGA.}}$ or $\underbrace{\Psi^{GS}_{\theta+2\pi} \perp \Psi^{GS}_\theta}_{\text{suggested by large-}N\text{ counting (witten)}}$.

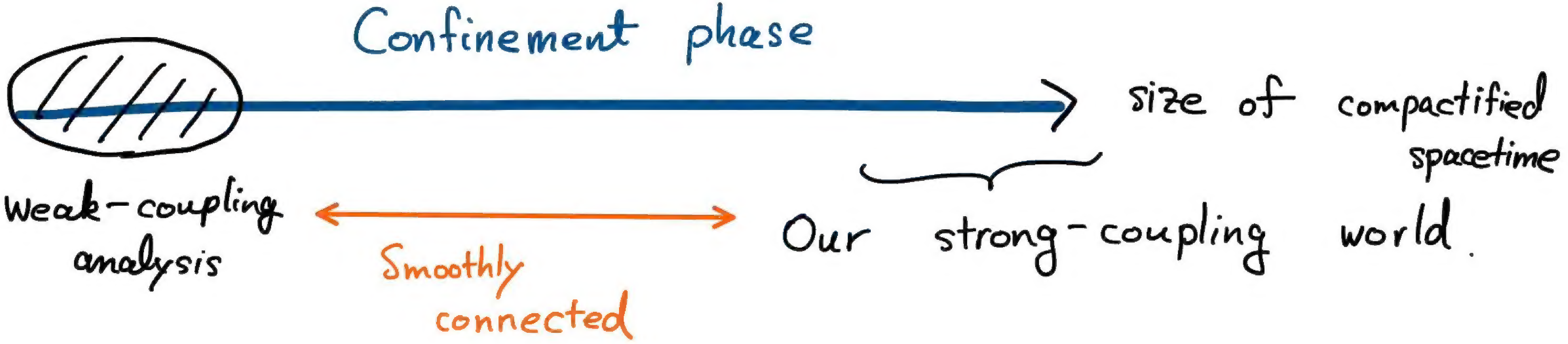
Gaiotto et al. '17 shows that if the vac. shows confinement

$e^{ik\,CS[A]}\,\Psi^{GS}_\theta[A]$ $(k=0,1,\cdots,N-1)$ are distinct as quantum phases.

$$\left(\Leftarrow\ Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B}\, Z_\theta[B]\right)$$

Adiabatic Continuity to Weak-Couplings

We would like to improve our understandings about microscopic mechanism of confinement and related phenomena.



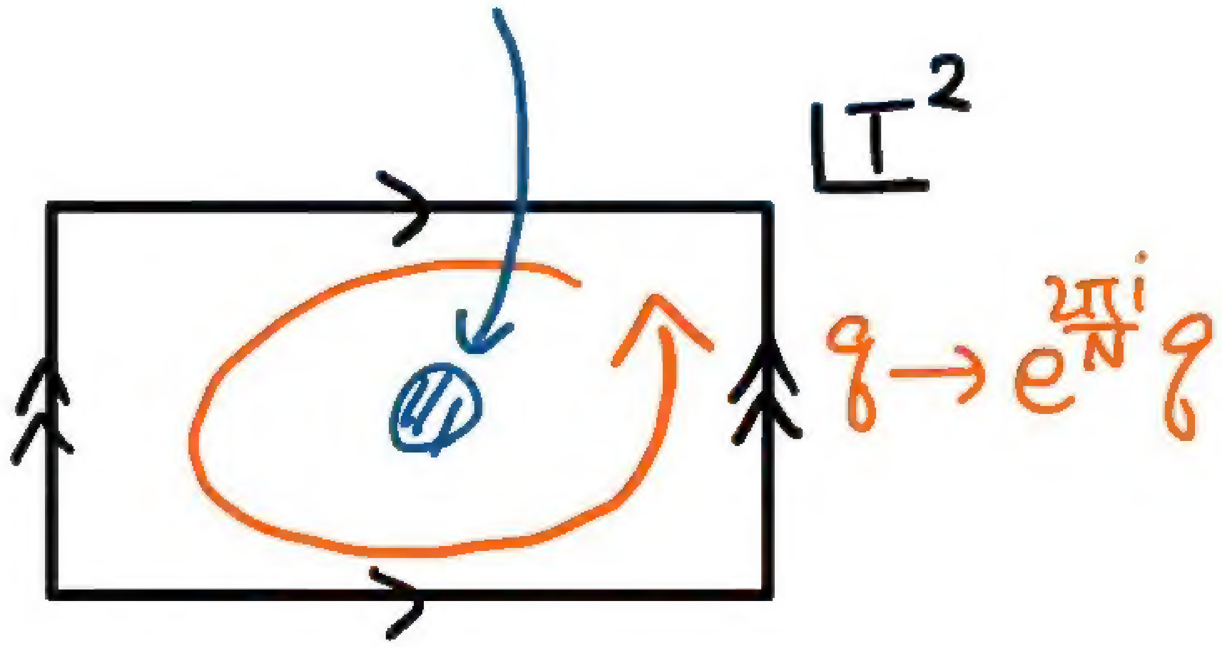
cf. · Twisted Eguchi-Kawai model (Gonzalez-Arroyo, Okawa)

· Double-trace deformed YM or YM theory w/ adjoint fermions } on $\mathbb{R}^3 \times S^1$ (Ünsal, ... 2007–)

Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2 \times T^2$

w/ $\boxed{\text{'t Hooft flux}}$



$\Longleftrightarrow$

Adiabatic Continuity

YM, QCD on $\mathbb{R}^4$

Weak-coupling description

via $\boxed{\text{Center Vortices}}$

Strong-couplings

(Related works: van Baal '84 (thesis), Yamazaki, Yonekura '17, Cox, Poppitz, Wandler '21 on $T^3 \times \mathbb{R}$)

Supporting evidence

For small $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux,

we can use the Dilute Gas Approximation w/ center vortices.

It predicts

- $\boxed{\text{(YM theory)}}$ $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)

- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

(Discrete chiral SSB)

- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\langle \mathrm{tr}_{cf}(\bar{\Psi})\, \mathrm{tr}_{cf}(\Psi) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

- $\boxed{\text{(QCD w/ } U(1)_B \text{ monopole flux)}}$ $S_{\mathrm{eff}} \sim \int \left(|dU|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \chi_{\mathrm{top}} (i \ln \det U - \theta)^2 \right)$

η' mass consistent with Witten-Veneziano formula

Role of 't Hooft flux on $\mathbb{R}^2 \times T^2$

- Preserve anomaly

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} Z_\theta[B] \quad \text{in 4d.}$$

$$\mathbb{Z}_N^{[1]} \xrightarrow{T^2\text{-compact.}} (\mathbb{Z}_N^{[1]})_{2d} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$$

$$B_{(4d)} = B_{2d} + A\wedge\frac{dx_3}{L} + A'\wedge\frac{dx_4}{L} + \overbrace{\frac{2\pi}{N} m \frac{dx_3\wedge dx_4}{L^2}}^{\text{'t Hooft flux}}$$

Then,

$$Z_{\theta+2\pi}[B_{2d}, A, A'] = Z_\theta[B_{2d}, A, A'] \, e^{i\boxed{m\int B_{2d}} - i\frac{N}{2\pi}\int A\wedge A'}$$

mixed anomaly between $(\mathbb{Z}_N^{[1]})_{2d}$ & $\theta\sim\theta+2\pi$.

- Preserve $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ center symmetry

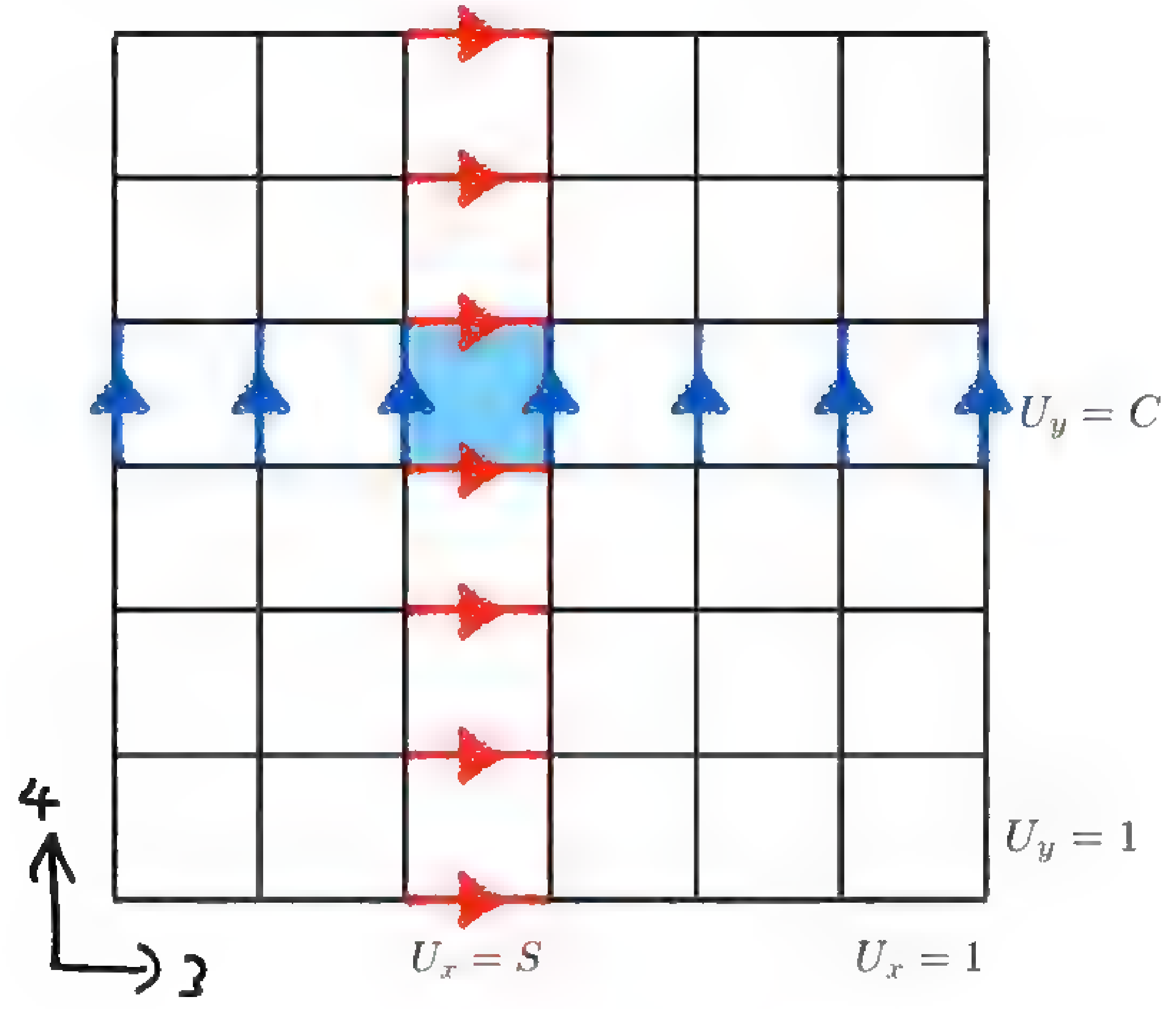
On classical vacuum, $\quad P_3 P_4 = P_4 P_3 \, e^{\frac{2\pi i}{N} m}.$

For $m \neq 0 \mod N$, $\quad \mathrm{tr}\, P_3 = \mathrm{tr}\, P_4 = 0.$ (cf. twisted Eguchi-Kawai)

(We set $m=1$ in the following)

- 4d instanton $\Longrightarrow$ N independent $\frac{1}{N}$-fractional instantons. (Montero, Gonzalez-Arroyo late '90s)

Classical configuration (= twist eater) & center symmetry



Lattice action

$$S_W[U_\ell, B] = -\frac{1}{g^2}\sum_P \left(e^{-\frac{2\pi i}{N}B_P}\,\mathrm{tr}[U_P] + e^{\frac{2\pi i}{N}}\,\mathrm{tr}[U_P^\dagger]\right).$$

$$B_P = \begin{cases} \frac{2\pi}{N} & \text{(for the plaquette indicated with light blue)} \\ 0 & \text{(otherwise)} \end{cases}$$

We can minimize this action by setting

$$U_\ell = \begin{cases} S = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \\ C = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \\ \mathbb{1} \end{cases}$$

$$\Rightarrow \quad P_3 = S, \quad P_4 = C.$$

This configuration completely preserves $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

Perturbative analysis of $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.

- 2d gluons are gapped. $\left(\sim \frac{1}{NL}\right)$

  $\Leftarrow$ Polyakov loops along T^2 are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S, \; P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N .$$

  Weak-coupling analysis is free from IR divergences.

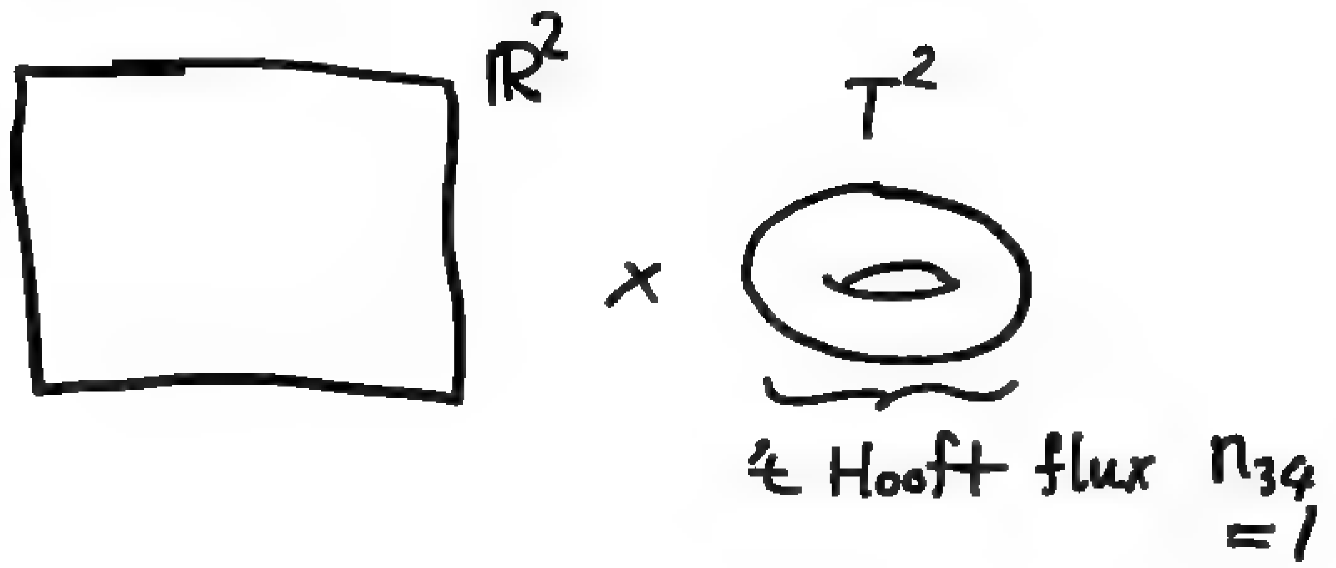
- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

  $\Downarrow$

  This issue is resolved by the semiclassics with center vortices.

Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by



$$\boxed{Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}}$$

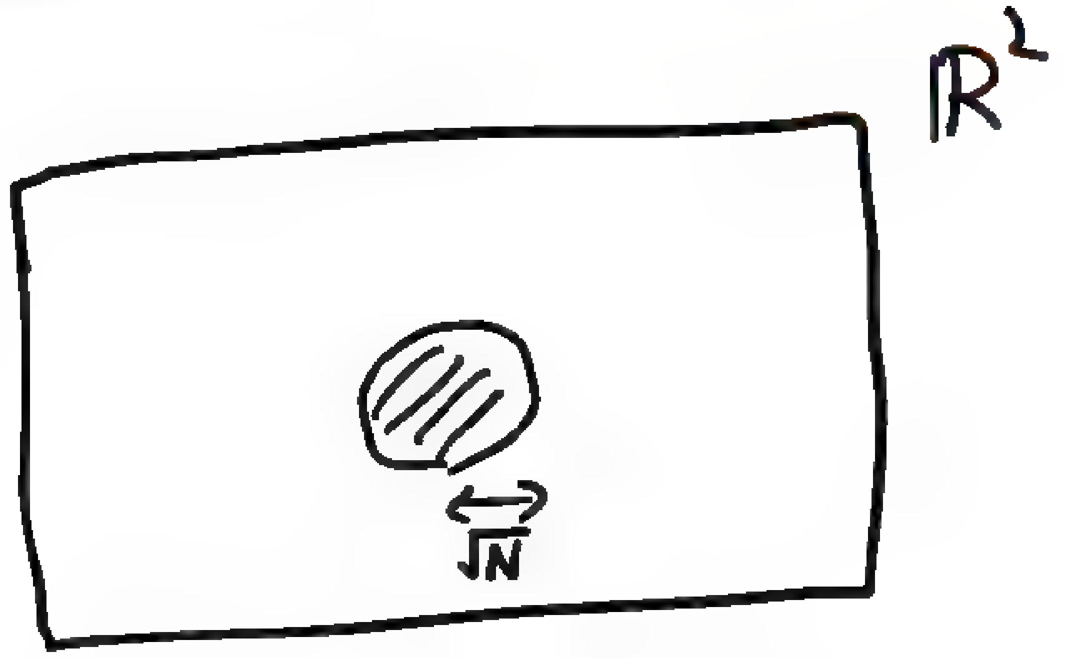
(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\varepsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$\boxed{S_{YM} = \frac{8\pi^2}{g^2} |Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex
or fractional instanton.



$$\left(\begin{array}{l} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{array}\right)$$

(cf. Garcia Perez, Gonzalez-Arroyo, '92, Itou '18)

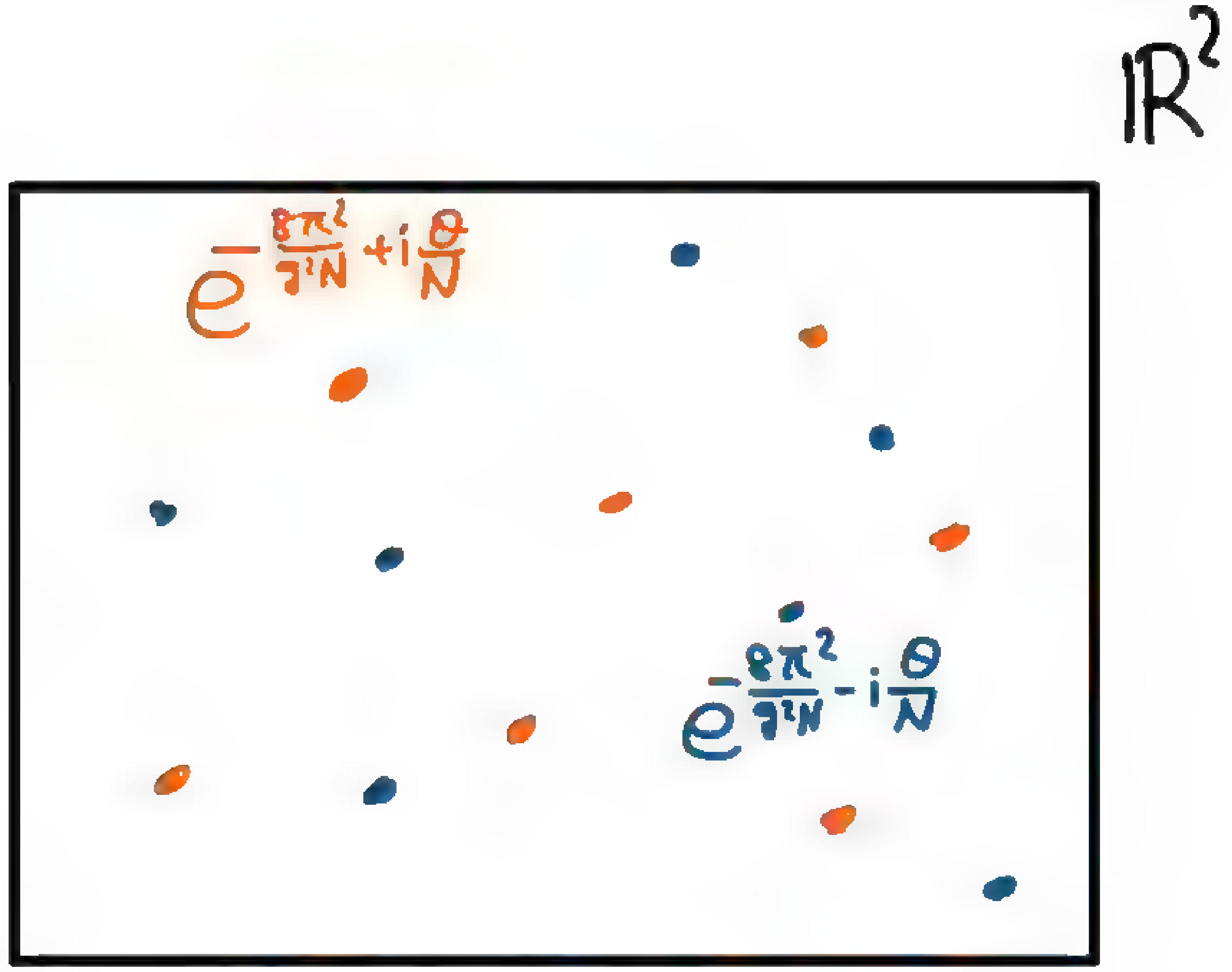
Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Rightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Rightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)



n : # of vortices

$\bar{n}$: # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

Partition function on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

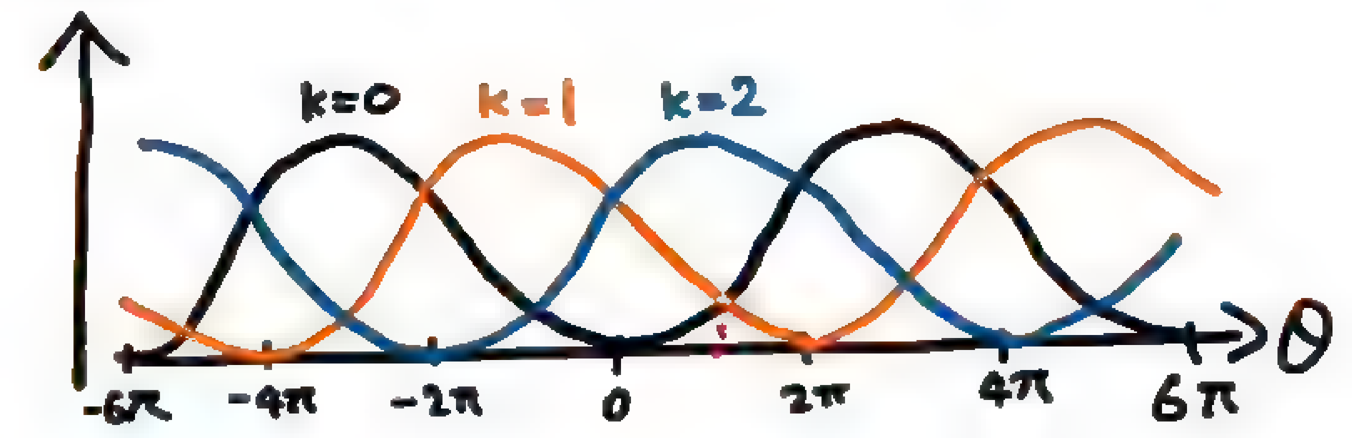
we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}_{\text{vortex}}\Big)^n \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\Big[-V\Big(\underbrace{-2K e^{-\frac{8\pi^2}{g^2N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)}_{E_k(\theta)\,:\,\text{Ground-state energy densities}}\Big)\Big]$$

$\Rightarrow \begin{cases} \bullet \text{ N-branch structure of ground states.} \\ \bullet \text{ Each branch has a fractional } \theta\text{-dependence.} \end{cases}$

$E_k(\theta)$ $(N=3)$



Partition function on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

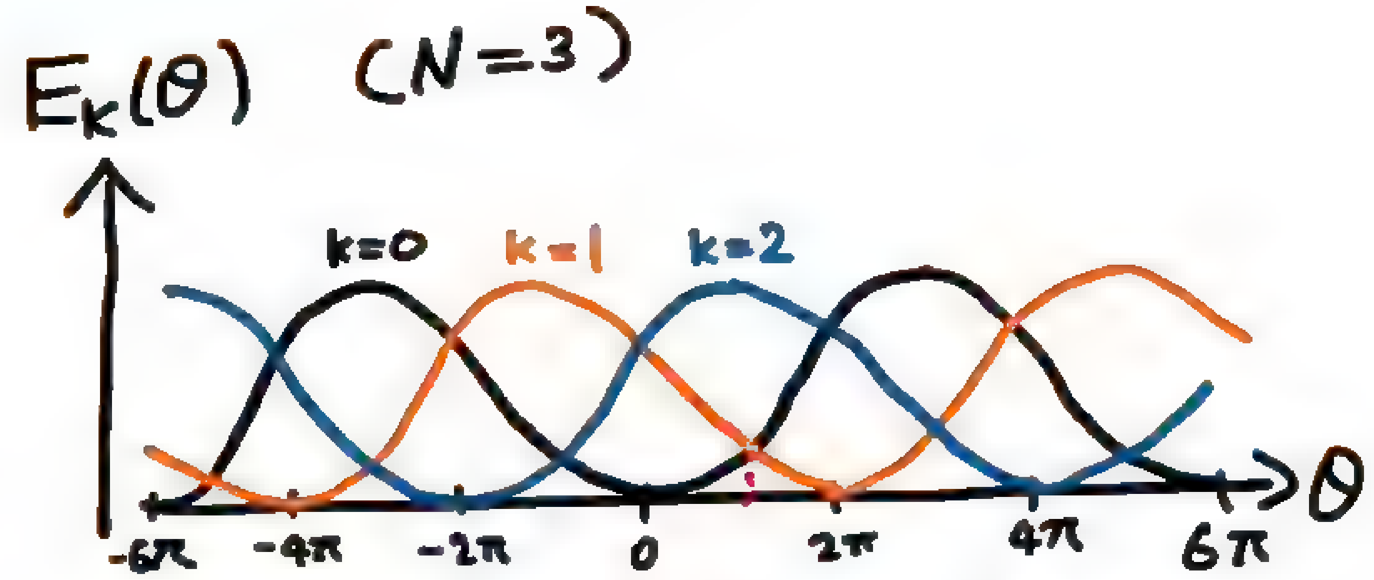
$$\left(\sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}(n-\bar{n})}\right)$$

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\;\bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}_{\text{vortex}}\Big)^{n} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\Big[-V\,\underline{\Big(-2K e^{-\frac{8\pi^2}{g^2N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)\Big)}\Big]$$

$E_k(\theta)$: Ground-state energy densities

$\Rightarrow$
- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Hamiltonian picture

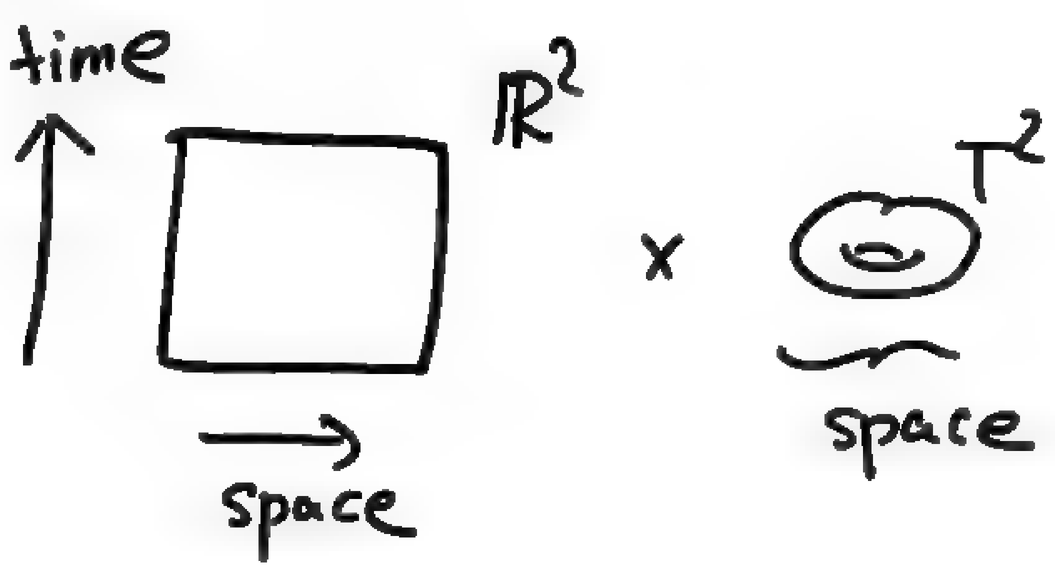
Using the dilute gas approximation of the YM path integral, we obtain

- N-branch structure of vacua
- Fractional θ-dependence $\quad E_k(\theta) \sim \Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$
- Confinement of Wilson loops for non-trivial N-alities.

To get a better understandings,

let us construct the wave-function of these confining vacua.

<u>Setup</u>

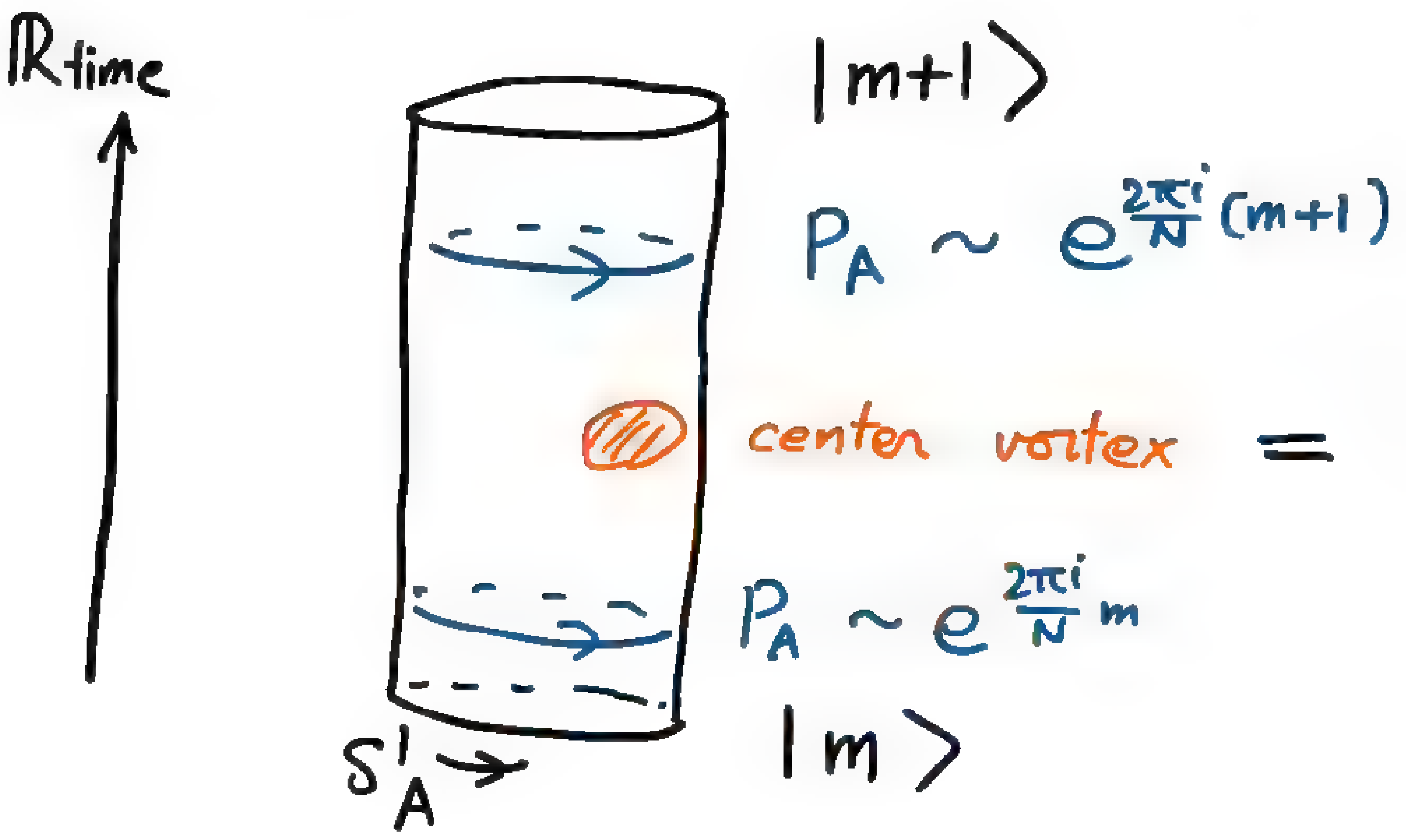


$$\Longrightarrow \quad \underset{\mathbb{R}_{\text{time}}}{\uparrow} \times \underbrace{S^1_A}_{\text{large}} \times \underbrace{S^1_B \times S^1_C}_{\text{small},\ n_{BC} = 1}$$

$\exists m$ s.t. $P_A = e^{\frac{2\pi i}{N} m} \mathbb{1}_N \leftrightarrow S^1_A$, $\quad P_B = S \leftrightarrow S^1_B$, $\quad P_C = C \leftrightarrow S^1_C$

(cf. Yamazaki, Yonekura '17 : $L_A, L_B \ll L_C \; (\ll \Lambda^{-1})$ w/ $n_{BC} = 1$)

Tunneling amplitude via center vortex



center vortex $=$ Tunneling from $|m\rangle$ to $|m+1\rangle$.

Within the one-center-vortex approximation,

$$\langle n|\exp(-T\hat{H}_{YM})|m\rangle$$

$$\sim \underbrace{\delta_{nm}}_{\text{No tunneling}} + T\cdot L_A\cdot K e^{-\frac{8\pi^2}{g^2N}}\Big(\underbrace{e^{i\frac{\theta}{N}}\delta_{n,m+1}}_{\text{tunneling via center vortex}} + \underbrace{e^{-i\frac{\theta}{N}}\delta_{n,m-1}}_{\text{tunneling via anti center vortex}}\Big)$$

Especially, $|m\rangle$ is not an eigenstate of $\hat{H}_{YM}$.

Eigenstate of $\exp(-T\hat{H}_{YM})$ and confining vacua

Let us consider the following wavefunction

$$|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} km} |m\rangle .$$

This is an eigenstate of $\exp(-T\hat{H}_{YM})$:

$$\exp(-T\hat{H}_{YM}) |\tilde{k}\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{n,m} e^{\frac{2\pi i}{N} km} |n\rangle \underbrace{\langle n| e^{-T\hat{H}_{YM}} |m\rangle}_{\sim \delta_{nm} + TL_A K e^{-\frac{8\pi^2}{g^2 N}} (e^{i\frac{\theta}{N}} \delta_{n,m+1} + e^{-i\frac{\theta}{N}} \delta_{n,m-1})}$$

$$\sim \exp\Big[-T\cdot L_A \cdot \underbrace{\Big(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\Big(\frac{\theta - 2\pi k}{N}\Big)\Big)}_{= E_k(\theta)}\Big] \cdot |\tilde{k}\rangle .$$

$\Rightarrow$ $|\tilde{k}\rangle$ give the wavefunction of confining vacua for $\mathbb{R}^2 \times \underbrace{T^2}_{\text{'t Hooft flux}}$.

We've seen that

YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux shares many important "qualitative" features of 4d confining vacua.

Can we extend this nice feature to 4d QCD w/ fund. quarks?

- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

(Hayashi, YT 2402 & YT, Ünsal 2201 (Sec. 4))

$U(1)_B$ vs $U(1)_q$

$$\text{Baryon numbers} = \frac{\text{quark numbers}}{N}$$

i.e. $$U(1)_B = U(1)_q / \mathbb{Z}_N .$$

Let us denote $U(1)_B, U(1)_q$ - gauge fields as A_B, A_q :

$$\frac{1}{N} A_B = A_q .$$

$$\left(\Leftarrow \quad A_{B\mu} J_B^\mu = A_{q\mu} J_q^\mu \quad \text{and} \quad J_B^\mu = \frac{1}{N} J_q^\mu \right)$$

However, this naive relation looks to be problematic when we consider monopoles :

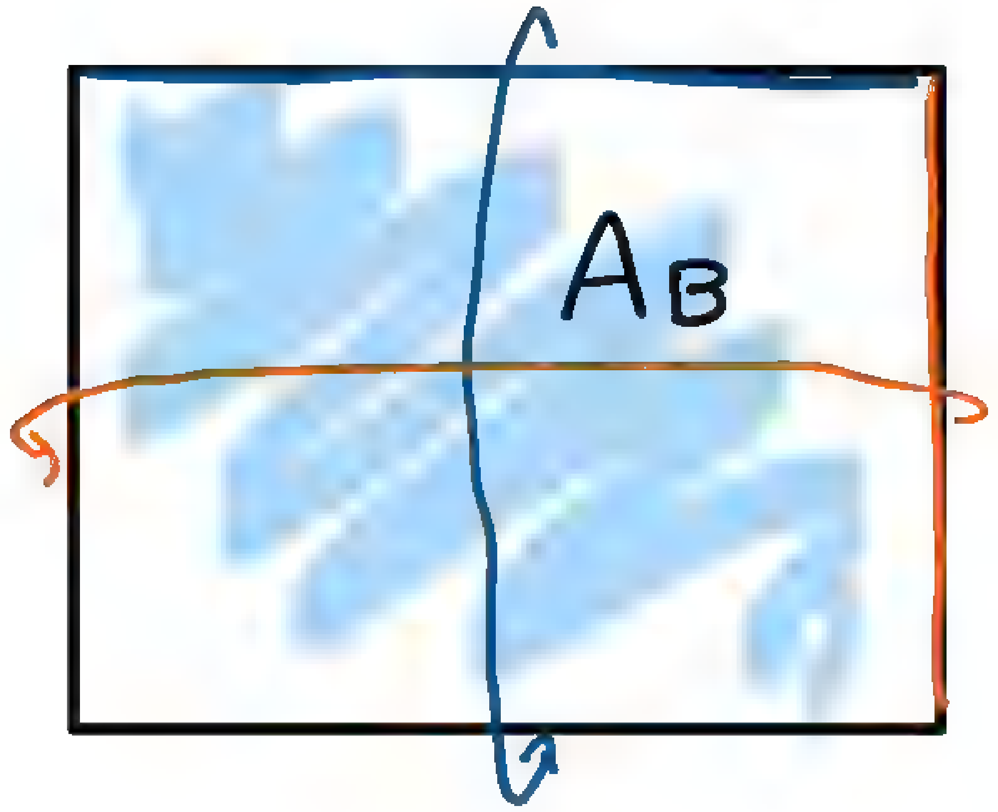
$U(1)_B$ monopole : $\int_{S^2} dA_B = 2\pi$. $\left(\Leftrightarrow \int_{S^2} dA_q = \frac{2\pi}{N} \right)$

$U(1)_q$ monopole : $\int_{S^2} dA_q = 2\pi$

normalization doesn't match.

$U(1)_B$ monopole violates Dirac quantization.

$U(1)_B$ monopole flux & 't Hooft flux on T^2



quark field

$$\begin{cases} \psi(L, y) = g_x^\dagger(y)\, e^{-i\frac{\phi_x(y)}{N}}\, \psi(0, y) \\ \psi(x, L) = \underbrace{g_y^\dagger(x)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_y(x)}{N}}}_{U(1)_B\text{-transition functions}}\, \psi(x, 0) \end{cases}$$

Cocycle condition

$$g_x^\dagger(L)\, g_y^\dagger(0)\, e^{-i\frac{1}{N}(\phi_x(L) + \phi_y(0))} = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{-i\frac{1}{N}(\phi_y(L) + \phi_x(0))}$$

$U(1)_B$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_x(L) - \phi_x(0)) - (\phi_y(L) - \phi_y(0))$$

$$\Longrightarrow \quad g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

Perturbative spectrum

Gauge fields : Same with $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

$\Rightarrow$ $\mathbb{Z}_N$ gauge field.

Quark fields : Solve the Dirac zero-mode eq.

$$\left[\gamma_3\left(\partial_3 + i\frac{1}{N}A_{B,3}\right) + \gamma_4\left(\partial_4 + i\frac{1}{N}A_{B,4}\right)\right]\psi = 0.$$

$\Rightarrow$ For each 4d fundamental Dirac fermion,

there is a 2d massless Dirac fermion.

For N_f - flavor massless QCD, we have 2d N_f Dirac fermions.

$$\xleftrightarrow[\text{bosonization}]{\text{non-Abelian}} \quad U(N_f)_1 \text{ WZW model} \quad \frac{1}{8\pi}\int_{M_2} \mathrm{tr}\left(dU^\dagger \wedge \star dU\right) + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}\left[(U^\dagger dU)^3\right]$$

w/ $U(N_f)$-valued field $U : M_2 \to U(N_f)$.

Center-vortex induced mass for η'

Let us construct the center-vortex vertex $\sim e^{-\frac{8\pi^2}{g^2 N}+i\frac{\theta}{N}}$.

$U(1)$ axial anomaly requires the spurious symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}, \quad \theta \to \theta + 2N_f\alpha.$$

In the bosonized description

$$U \to e^{-2i\alpha} U.$$

$$\Rightarrow \quad e^{-\frac{8\pi^2}{g^2 N}+i\frac{\theta}{N}} \cdot (\det U)^{\frac{1}{N}}$$

$\frac{1}{N}$ fractionalization of Kobayashi-Maskawa-'t Hooft vertex

satisfies

- $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry,
- $U(1)$ axial anomaly relation.

$$\Rightarrow \quad \Delta S_{\text{center-vortex}} \sim \underbrace{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}_{\eta' \text{ mass}} \cdot \cos\left(\frac{\overbrace{i\ln(\det U)}^{\eta'} - \theta}{N}\right)$$

On η' periodicity

$$\eta'\text{-mass} \sim -\cos\left(\frac{\eta'-\theta}{N}\right).$$

But the chiral effective field is $\left(U(N_f) = \dfrac{U(1)\times SU(N_f)}{\mathbb{Z}_{N_f}} \right)$

$$U = \underbrace{e^{i\eta'/N_f}}_{U(1)\text{-factor}} \cdot \underbrace{V}_{SU(N_f)} \quad \text{with } \left(\eta'+2\pi, e^{-\frac{2\pi i}{N_f}}V\right)\sim(\eta', V).$$

$\Rightarrow$ The η'-potential does not respect the naive periodicity.

More careful treatment gives

$\mathbb{Z}_N$ label for YM confineming vacua

$$\text{DIGA of } \tfrac{1}{N}\text{-fractional KMT vertex} = \sum_{k=0}^{N-1} e^{-\cdots\cos\left(\frac{\eta'+2\pi k-\theta}{N}\right)}.$$

η' extends its periodicity as

$$\begin{cases} \bullet\ k+N \sim k \\ \bullet\ (\eta', V, k+1) \sim (\eta'+2\pi, V, k) \\ \bullet\ (\eta'+2\pi, V, k) \sim (\eta', e^{\frac{2\pi i}{N_f}}V, k) \end{cases} \xRightarrow{\text{Eliminate } k} \bullet\ (\eta'+2\pi N, V)\sim\left(\eta', e^{\frac{2\pi i N}{N_f}} V\right).$$

Baryon - color - flavor anomaly

Assuming the extended periodicity $\eta' \sim \eta' + 2\pi N_c$ also in 4d,

we can reproduce the BCF anomaly from chiral Lagrangian.

Vector-like sym. of QCD : $\dfrac{SU(N_f) \times U(1)_q}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}}$ $\frown (A_f, A_q)$: 1-form gauge field

$\smile (B_c, B_f)$: 2-form gauge field

$$\mathcal{L}_{QCD,\theta+2\pi} = \mathcal{L}_{QCD,\theta} + \underbrace{\frac{2\pi i}{N_c} \cdot \frac{1}{8\pi^2} \int (dA_B)^2}_{\text{potential anomalous term.}} \qquad dA_B = N_c\, dA_q + B_c$$

(Gaiotto, Komargodski, Seiberg '17
YT, Kikuchi '17, YT '18)

(* This anomaly is fake when $\gcd(N_c, N_f) = 1$.
When $\gcd(N_c, N_f) \neq 1$, this is genuine as no local counter terms can eliminate it.)

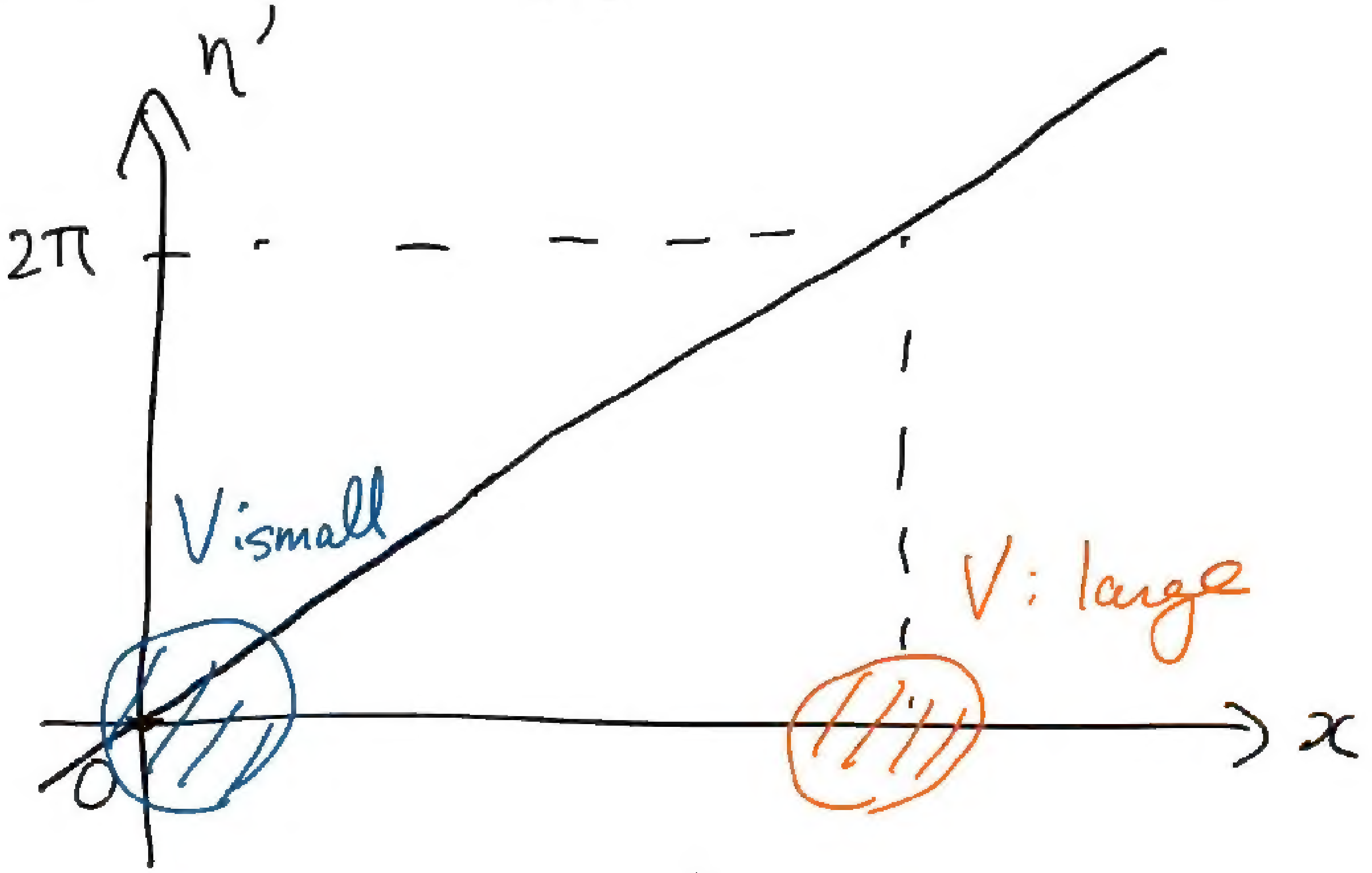
$$\Leftarrow \quad \Gamma_{WZW} \supset \frac{i}{24\pi^2} \int_{M_4} A_B \wedge \mathrm{tr}((U^{-1}dU)^3) + i \frac{1}{8\pi^2 N_c} \int_{M_4} \eta' \wedge (dA_B)^2 .$$

This $\frac{1}{N_c}$ is allowed as $\eta' \sim \eta' + 2\pi \underline{\underline{N_c}}$.

Physical (?) applications

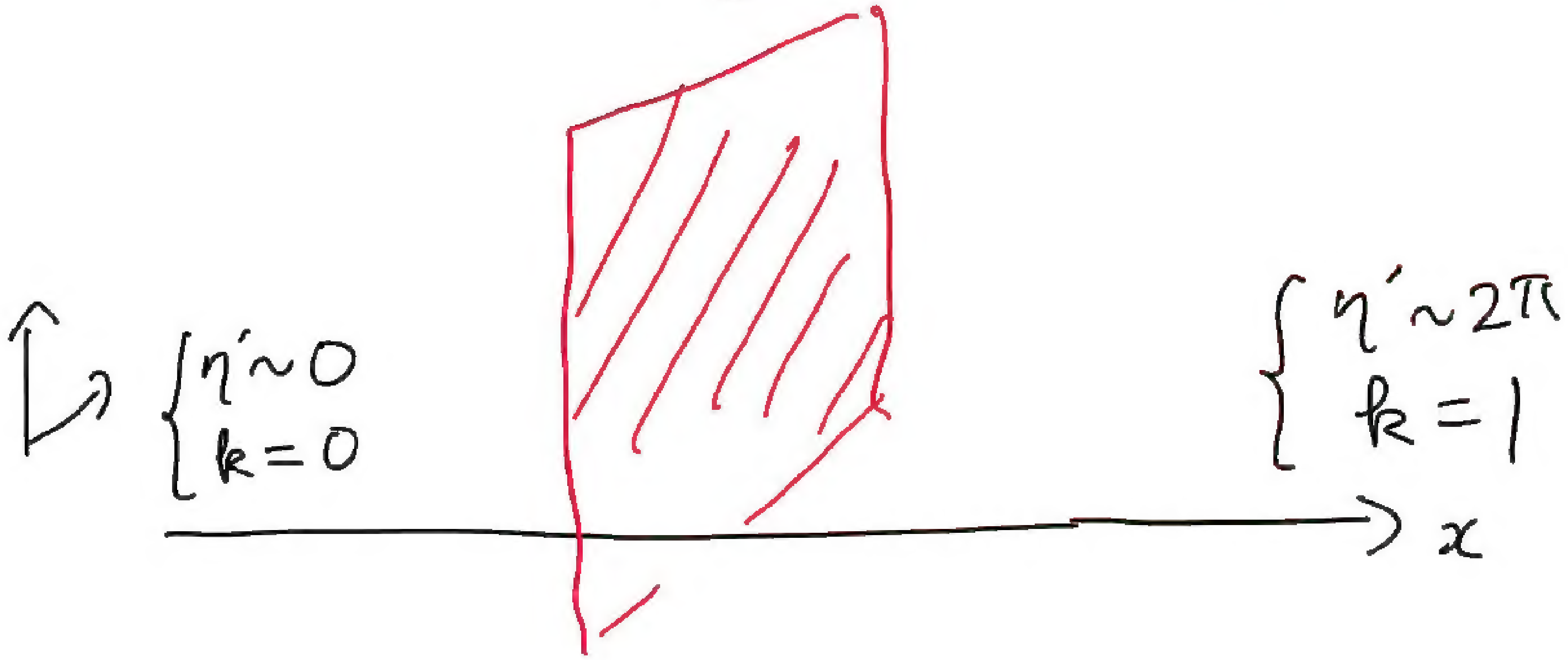
Assume we could apply an external potential so that $\eta' \nearrow$.

$$V \sim - \cos\left(\frac{\eta'}{N}\right)$$



$$\Downarrow$$

$\eta' \sim \pi$, DW w/ k jump : $SU(N)_1$ CS theory on the wall



(cf. Komargodski's Quantum Hall Droplet Baryon)

Summary

- Topological methods in QFT are highly developed.

  $\rightsquigarrow$ Many nontrivial constraints on dynamics of QFTs.

  But, they do not tell the concrete microscopic dynamics.

- 4d YM, QCD on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux

  $\xrightarrow{\text{(Adiabatic continuity assumption)}}$ Concrete derivation of $\begin{cases} \bullet \text{ multi-branch str. of confining vacua} \\ \bullet \text{ chiral Lagrangian of QCD} \\ \bullet \text{ new insight of } \eta'. \end{cases}$

- Applications to other 4d confining gauge theories.
  - Orbifold/orientifold equivalence for $\begin{cases} \text{2-index QCD (YT, Ünsal 2205.11339)} \\ SU(N)\times SU(N) \text{ bifund. QCD (Hayashi, YT, Watanabe 2307.13954)} \end{cases}$
  - Non-supersymmetric duality cascade $\begin{pmatrix} \text{Hayashi, YT, Watanabe (ongoing)} \\ \text{Karasik, Komargodski 1904.09551} \end{pmatrix}$